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Measurements

If you have a smart project, you can say "I'm an engineer"

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Lecture 1

<u>Staff boarder</u>

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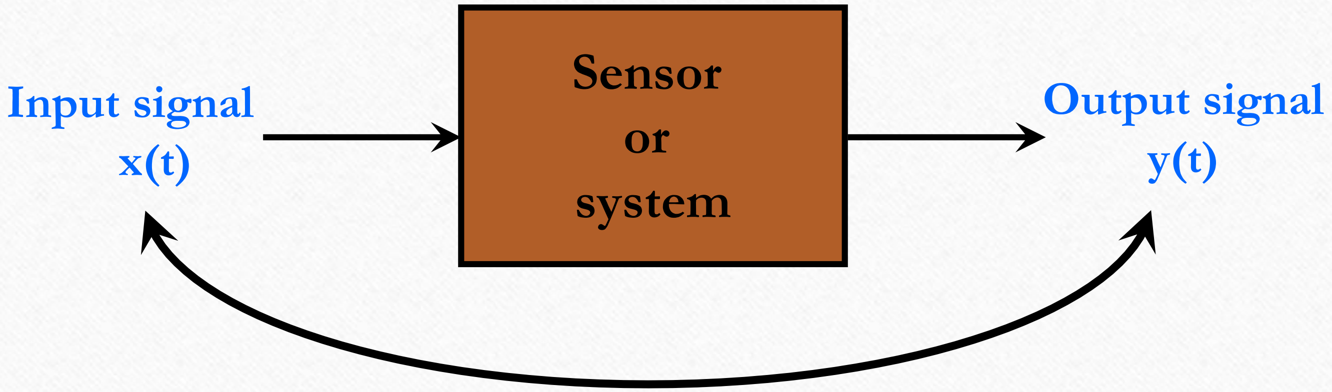
Measurements

- **Lecture aims:**
  - Solve simple problems using the maximum principle
  - Formulate advanced problems for numerical solution

Dynamic Characteristics

Dynamic characteristics tell us about how well a sensor responds to changes in its input. For dynamic signals, the sensor or the measurement system must be able to respond fast enough to keep up with the input signals.



In many situations, we must use y(t) to infer x(t), therefore a qualitative understanding of the operation that the sensor or measurement system performs is imperative to understanding the input signal correctly.

General Model For A Measurement System

n^{th} Order ordinary linear differential equation with constant coefficient

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = \underbrace{b_m \frac{d^m x(t)}{dt^m} + b_{m-1}\frac{d^{m-1} x(t)}{dt^{m-1}} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)}_{F(t)\ =\ \text{forcing function}}$$

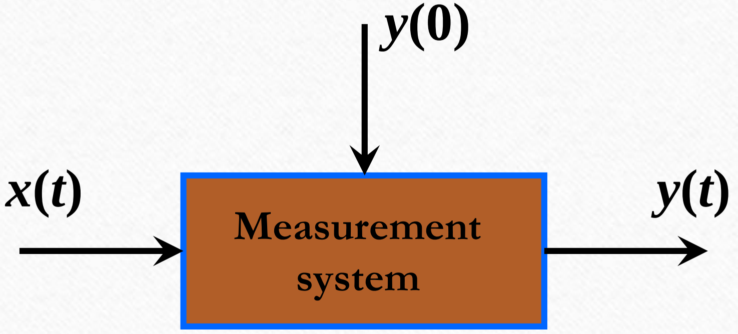
Where $m \leq n$

$y(t)$ = output from the system

$x(t)$ = input to the system

t = time

a's and b's = system physical parameters, assumed constant



The solution $y(t) = y_{ocf} + y_{opi}$

Where y_{ocf} = complementary-function part of solution

y_{opi} = particular-integral part of solution

Complementary-Function Solution

The solution y_{ocf} is obtained by calculating the n roots of the algebraic *characteristic equation*

Characteristic equation $$a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0 = 0$$

Roots of the characteristic equation: $$D = s_1, s_2, \ldots, s_n$$

Complementary-function solution:

1. Real roots, unrepeated: $$Ce^{st}$$
2. Real roots, repeated:
   each root s which appear p times $$\left(C_0 + C_1 t + C_2 t^2 + \ldots + C_{p-1} t^{p-1}\right) e^{st}$$
3. Complex roots, unrepeated:
   the complex form: $a \pm ib$ $$Ce^{at} \sin(bt + \phi)$$
4. Complex roots, repeated:
   each pair of complex root which appear p times $$[C_0 \sin(bt + \phi_0) + C_1 t \sin(bt + \phi_1) + C_2 t^2 \sin(bt + \phi_2) + \ldots + C_{p-1} t^{p-1} \sin(bt + \phi_{p-1})] e^{at}$$

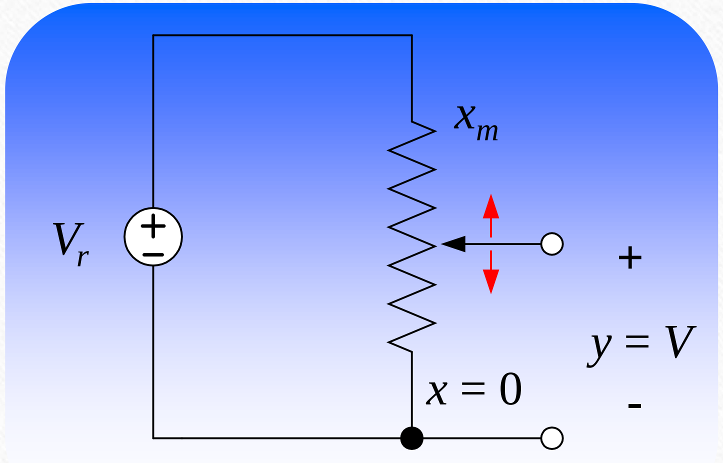
Zero-order Systems

All the a's and b's other than a_0 and b_0 are zero.

$$a_0 y(t) = b_0 x(t) \longrightarrow y(t) = Kx(t)$$

where K = static sensitivity = b_0/a_0

The behavior is characterized by its static sensitivity, K and remains constant regardless of input frequency (ideal dynamic characteristic).



$$V = V_r \cdot \frac{x}{x_m} \text{ here, } K = V_r / x_m$$

Where $0 \le x \le x_m$ and V_r is a reference voltage

A linear potentiometer used as position sensor is a zero-order sensor.

First-Order Systems

All the a's and b's other than a_1, a_0 and b_0 are zero.

$$a_1 \frac{dy(t)}{dt} + a_0 = b_0 x(t)$$

$$\boxed{\tau \frac{dy(t)}{dt} + y(t) = Kx(t)} \quad \Longleftrightarrow \quad \boxed{\frac{y}{x}(D) = \frac{K}{\tau D + 1}}$$

Where $K = b_0/a_0$ is the static sensitivity

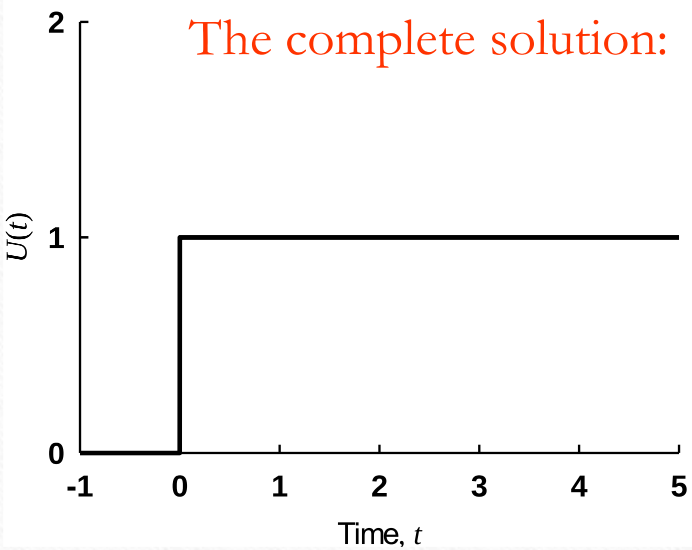
$\tau = a_1/a_0$ is the system's time constant (dimension of time)

First-Order Systems: Step Response

Assume for $t < 0$, $y = y_0$, at time $= 0$ the input quantity, x increases instantly by an amount A. Therefore $t > 0$

$$x(t) = AU(t) = \begin{cases} 0 & t \leq 0 \\ A & t > 0 \end{cases}$$

$$\tau \frac{dy(t)}{dt} + y(t) = KAU(t)$$



The complete solution:

$$\boxed{y(t) = Ce^{-t/\tau} + KA}$$

$$\underbrace{Ce^{-t/\tau}}_{y_{ocf}} \quad \underbrace{KA}_{y_{opi}}$$

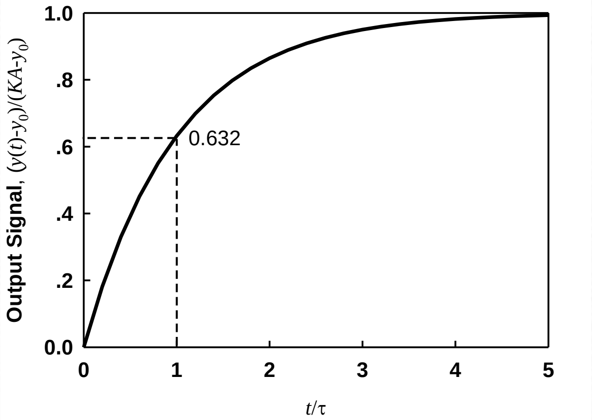
Transient response — **Steady state response**

Applying the initial condition, we get $C = y_0 - KA$, thus gives

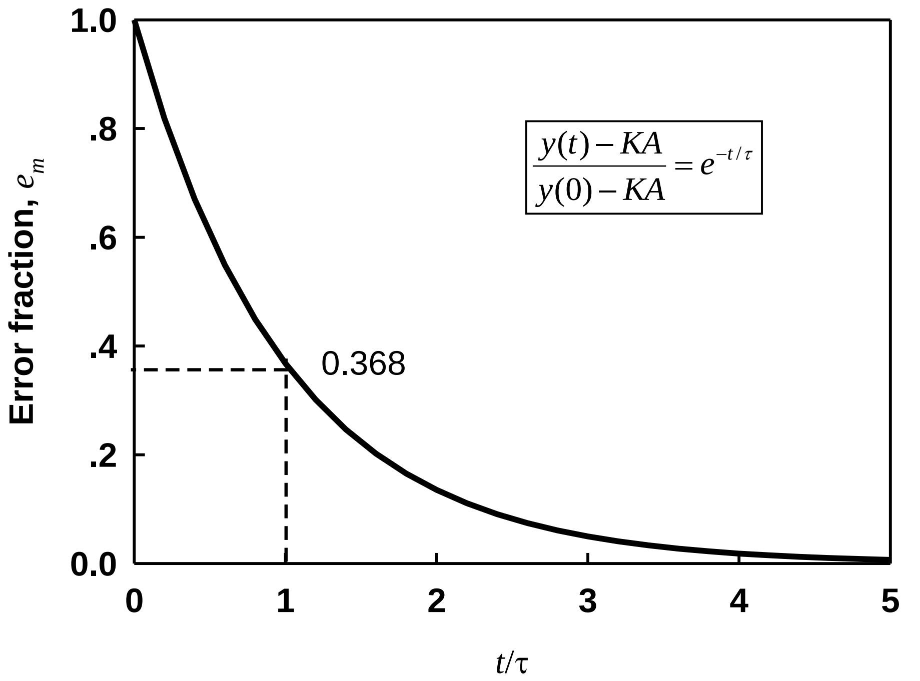
$$\boxed{y(t) = KA + (y_0 - KA)e^{-t/\tau}}$$

First-Order Systems: Step Response

Here, we define the term error fraction as $e_m(t) = \frac{y(t) - KA}{y_0 - KA} = \frac{y(t) - y(\infty)}{y(0) - y(\infty)} = e^{-t/\tau}$



$$\frac{y(t) - y_0}{KA - y_0} = 1 - e^{-t/\tau}$$



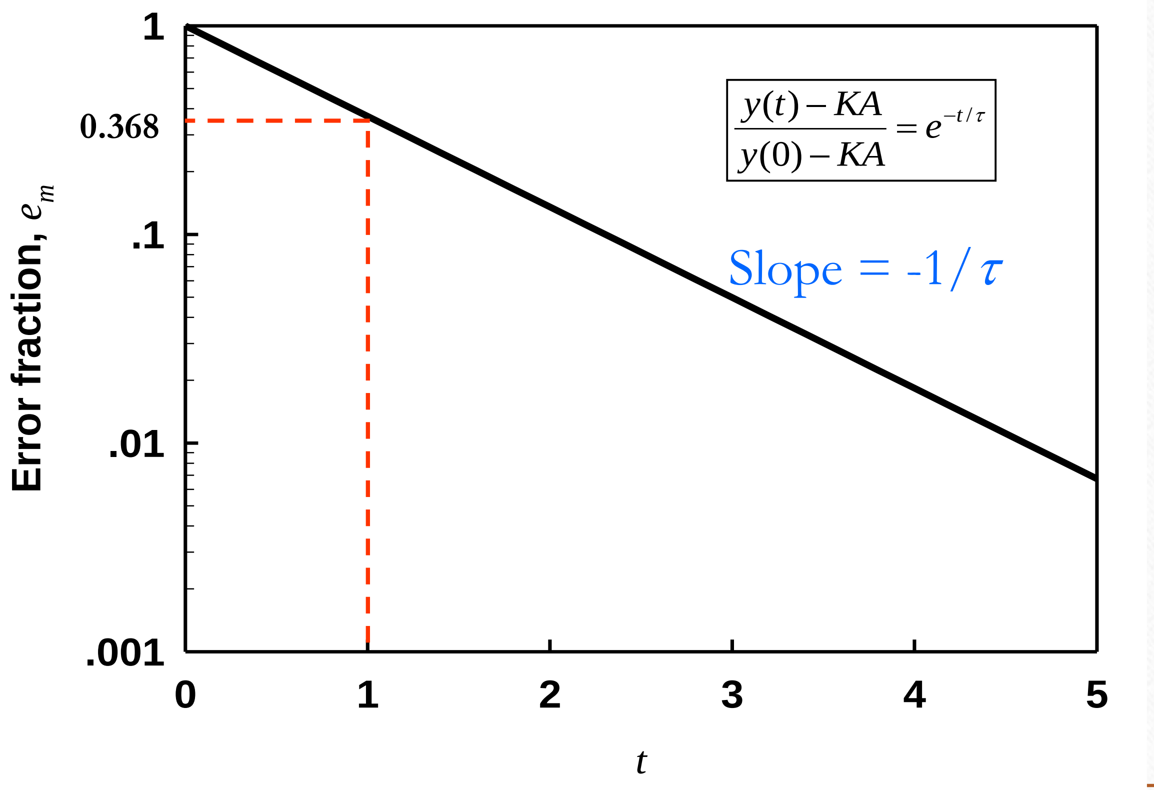
$$\frac{y(t) - KA}{y(0) - KA} = e^{-t/\tau}$$

Non-dimensional step response of first-order instrument

Determination of Time constant

$$e_m = \frac{y(t) - KA}{y(0) - KA} = e^{-t/\tau}$$

$$\ln e_m = 2.3 \log e_m = -\frac{t}{\tau}$$



First-Order Systems: Ramp Response

Assume that at initial condition, both y and x = 0, at time = 0, the input quantity start to change at a constant rate $\dot{q}_{is}$ Thus, we have

$$x(t) = \begin{cases} 0 & t \le 0 \\ \dot{q}_{is} t & t > 0 \end{cases}$$

Therefore

$$\tau \frac{dy(t)}{dt} + y(t) = K\dot{q}_{is} t U(t)$$

The complete solution:

$$y(t) = \underbrace{Ce^{-t/\tau}}_{\text{Transient response}} + \underbrace{K\dot{q}_{is}(t-\tau)}_{\text{Steady state response}}$$

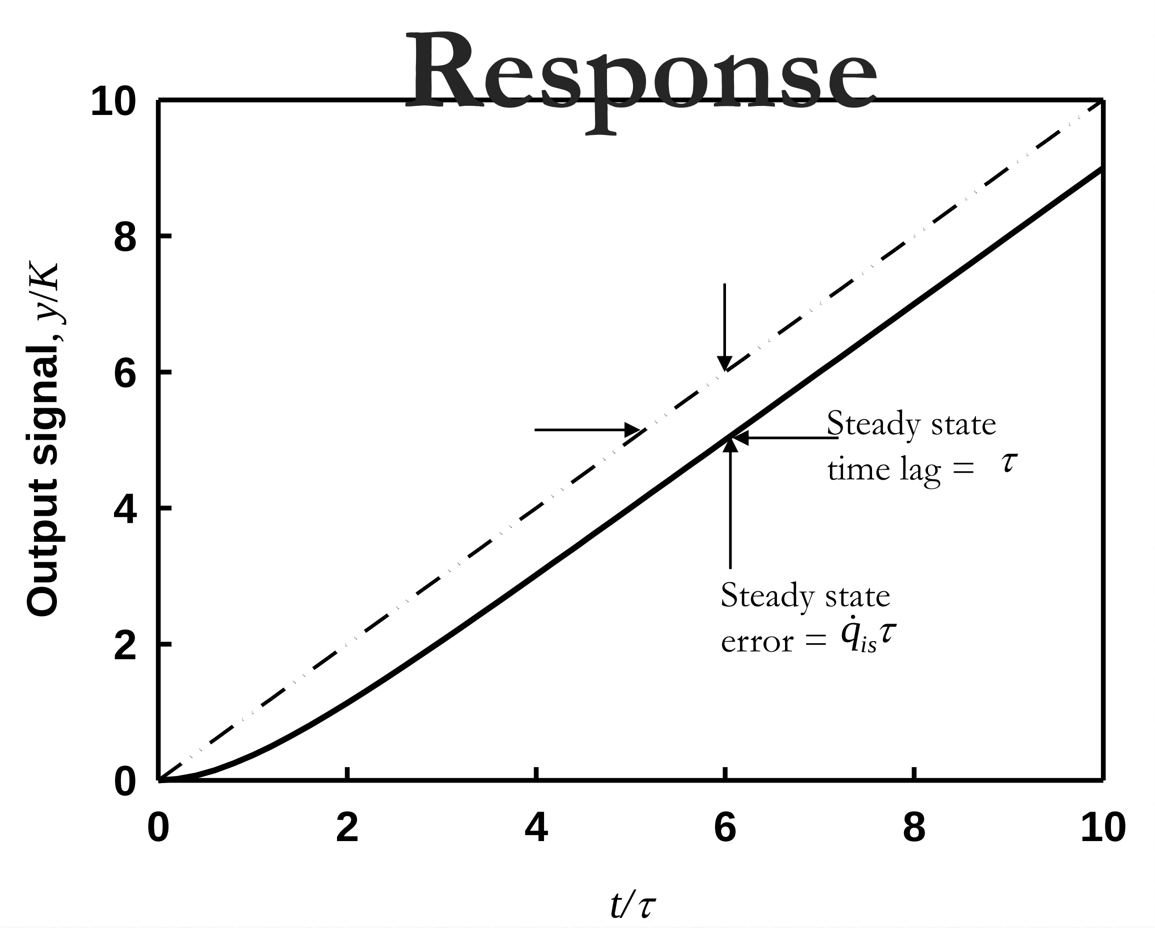
Applying the initial condition, gives

$$\boxed{y(t) = K\dot{q}_{is}(\tau e^{-t/\tau} + t - \tau)}$$

Measurement error

$$\boxed{e_m = x(t) - \frac{y(t)}{K} = \underbrace{-\dot{q}_{is}\tau e^{-t/\tau}}_{\text{Transient error}} + \underbrace{\dot{q}_{is}\tau}_{\text{Steady state error}}}$$

First-Order Instrument: Ramp Response



Non-dimensional ramp response of first-order instrument

First-Order Systems: Frequency Response

From the response of first-order system to sinusoidal inputs, we have

$$x(t) = A\sin\omega t$$

$$\tau\frac{dy}{dt} + y = KA\sin\omega t \quad \Longleftrightarrow \quad (\tau D + 1)y(t) = KA\sin\omega t$$

The complete solution:

$$y(t) = \underbrace{Ce^{-t/\tau}}_{\text{Transient response}} + \underbrace{\frac{KA}{\sqrt{1+(\omega\tau)^2}}\sin\left(\omega t - \tan^{-1}\omega\tau\right)}_{\text{Steady state response}}$$

Steady state response = Frequency response

If we do interest in only steady state response of the system, we can write the equation in general form

$$y(t) = Ce^{-t/\tau} + B(\omega)\sin[\omega t + \phi(\omega)]$$

$$B(\omega) = \frac{KA}{\left[1+(\omega\tau)^2\right]^{1/2}}$$

$$\phi(\omega) = -\tan^{-1}\omega\tau$$

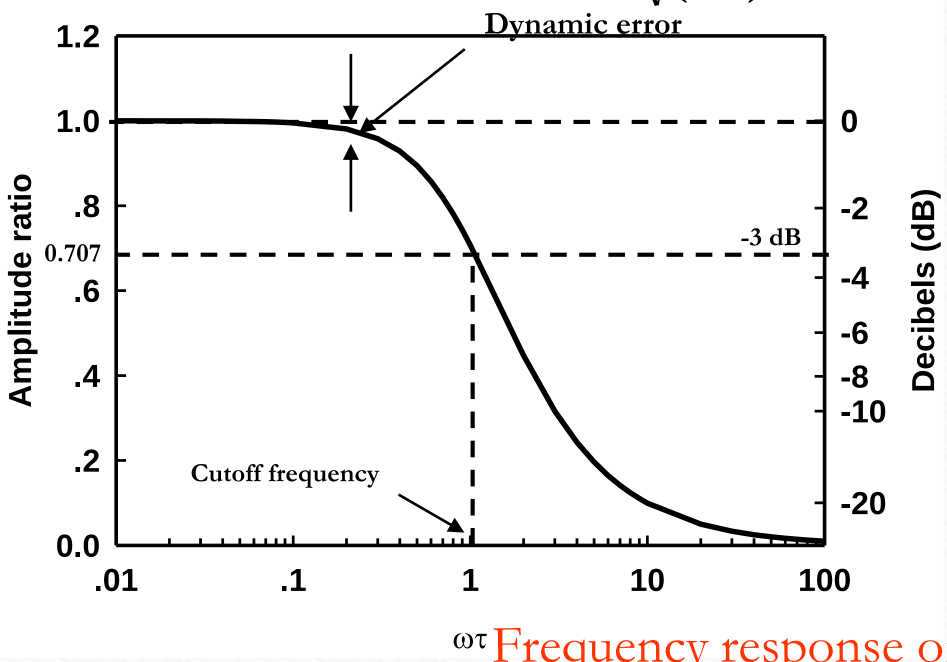
Where $B(\omega)$ = amplitude of the steady state response and $\phi(\omega)$ = phase shift

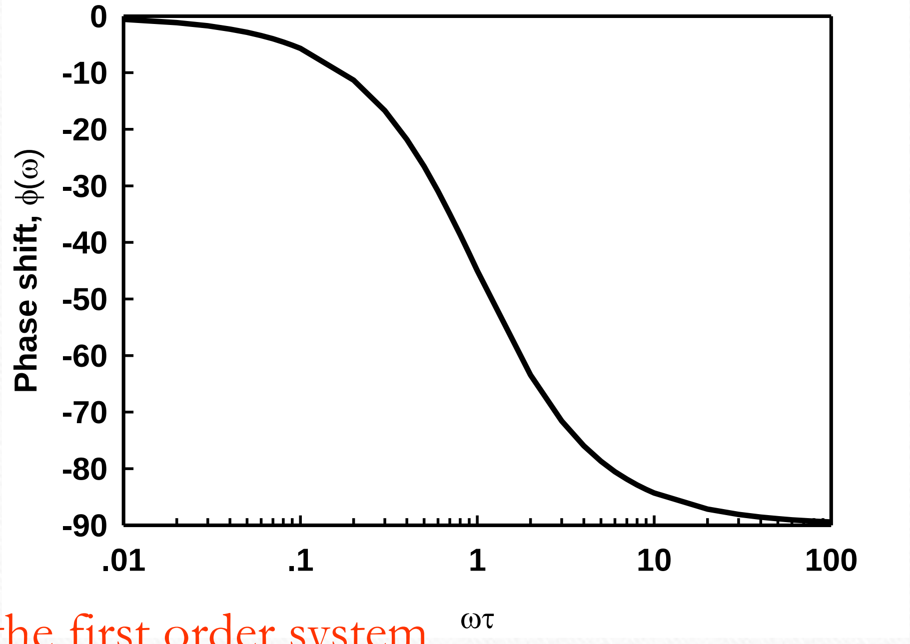
First-Order Instrument: Frequency Response

$$M(\omega) = \frac{B}{KA} = \frac{1}{\left[1+(\omega\tau)^2\right]^{1/2}}$$

The amplitude ratio $M(\omega) = \frac{1}{\sqrt{(\omega\tau)^2+1}}$

The phase angle is $\phi(\omega) = -\tan^{-1}(\omega\tau)$





Frequency response of the first order system

Dynamic error, $\delta(\omega) = M(\omega)$: a measure of an inability of a system to adequately reconstruct the amplitude of the input for a particular frequency

Dynamic Characteristics

Frequency Response describe how the ratio of output and input changes with the input frequency. (sinusoidal input)

Dynamic error, $\delta(\omega) = 1 - M(\omega)$ a measure of the inability of a system or sensor to adequately reconstruct the amplitude of the input for a particular frequency

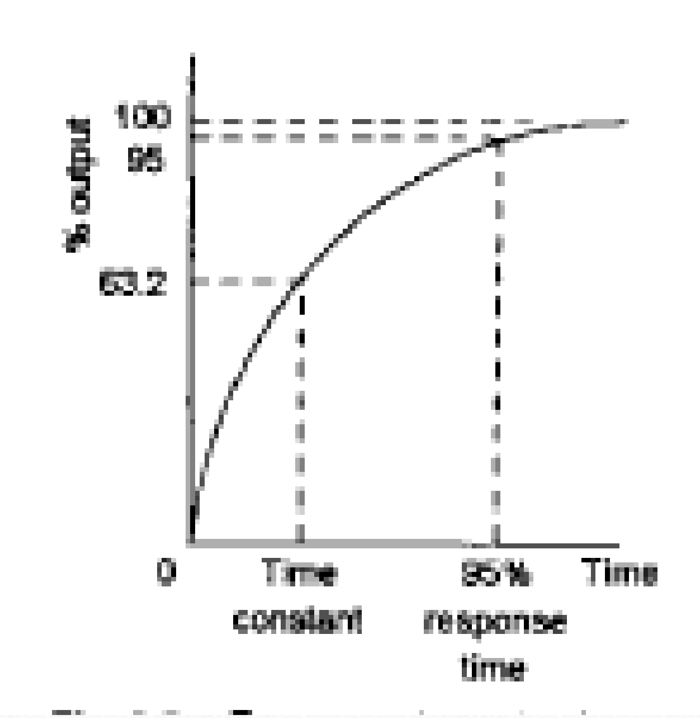
Bandwidth the frequency band over which $M(\omega) \geq 0.707$ (-3 dB in decibel unit)

Cutoff frequency: the frequency at which the system response has fallen to 0.707 (-3 dB) of the stable low frequency.

$$t_r \approx \frac{0.35}{f_c}$$

Static & Dynamic Characteristics

- Response time: 95% of final value for step input
- Time constant : 63.2% ($1-e^{-1}$) of final value specified percentage of steady state output
- Settling time: This is the time taken for the output to settle to within some percentage, e.g. 2% of the s.s. value



Measurement System Behavior

- Each measurement system will respond differently to different input signals.
- Therefore, a particular measurement system may or may not be suitable for making particular measurements.
- However, a measurement system will always output something, regardless of how poorly this output actually represents the measured signal.
- It is important to understand how measurement systems work.

Measurement System Behavior

Review…

Static measurement – measured signal does not change (the length of a piece of string), only the amplitude is important.

Dynamic measurement – measured signal changes as a function of time (wind speed), the amplitude (as a function of time), frequency, general waveform, etc. all become important.

The measurement system must respond fast enough to capture all the important aspects of the measured signal.

Measurement System Behavior

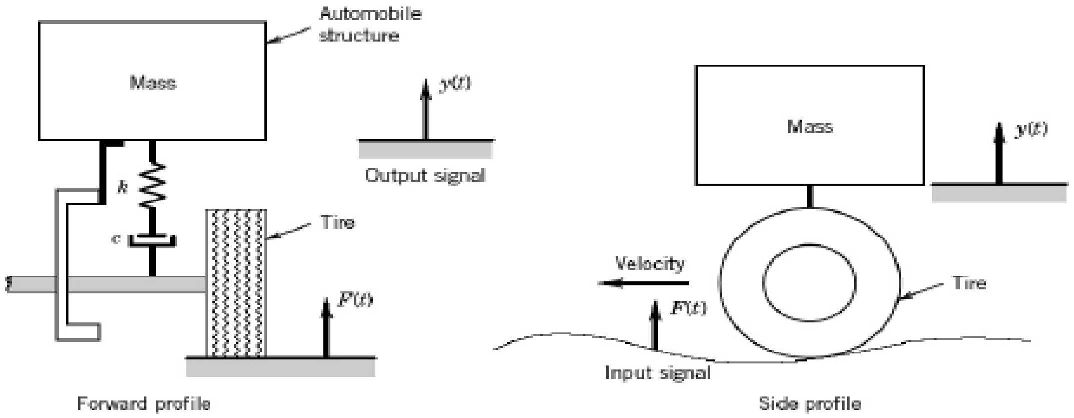
Example #1 – body temperature thermometer.

1. Initial condition – room temperature
2. Insert into (or hold close to) the body
3. The response of the thermometer takes time. The temp displayed by the thermometer slowly changes from room temp to body temp.
4. The temp of the body is not changing, the “measurement system” is responding to a “step change” in the temp.

Note: Body temp is constant (static), but the measurement system experiences a sudden change (dynamic) and must respond.

Measurement System Behavior

Example #2 – Automobile ride quality (suspension system).

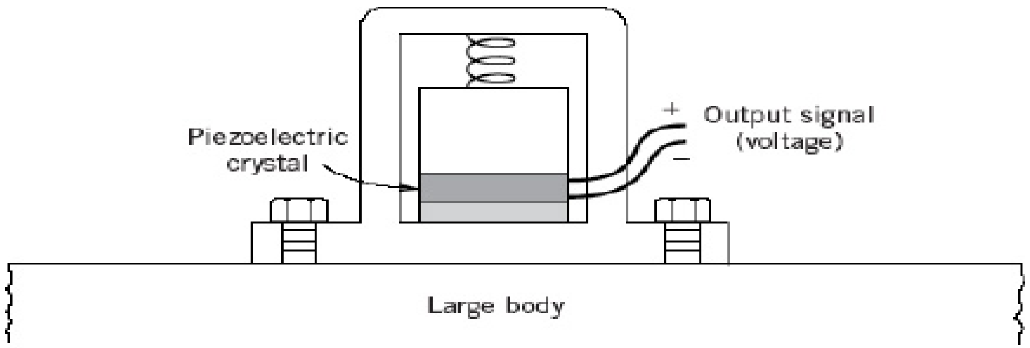


Suspension system – road and car interact to provide ride quality.
Input signal and the measurement system interact to form an output signal.

Measurement System Behavior

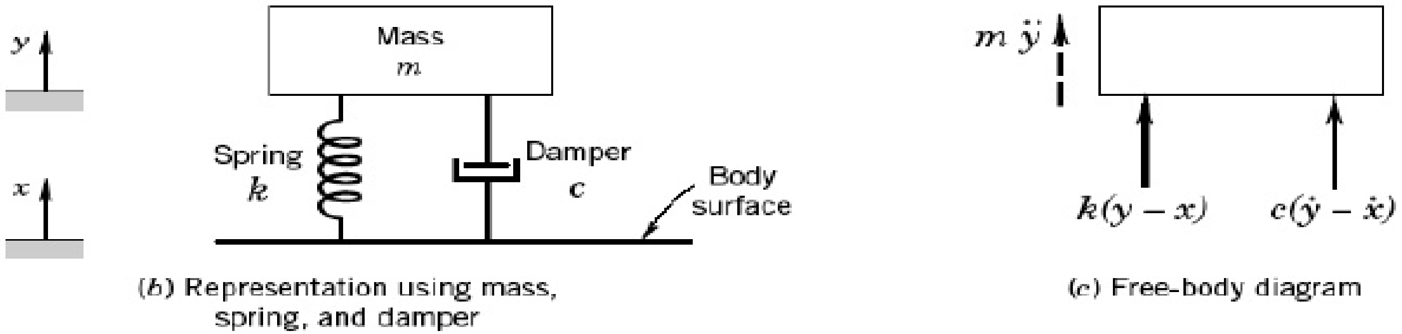
Most measurement systems can be represented as linear, ordinary differential equations.

Example – Piezoelectric Accelerometer



(a) Piezoelectric accelerometer attached to large body

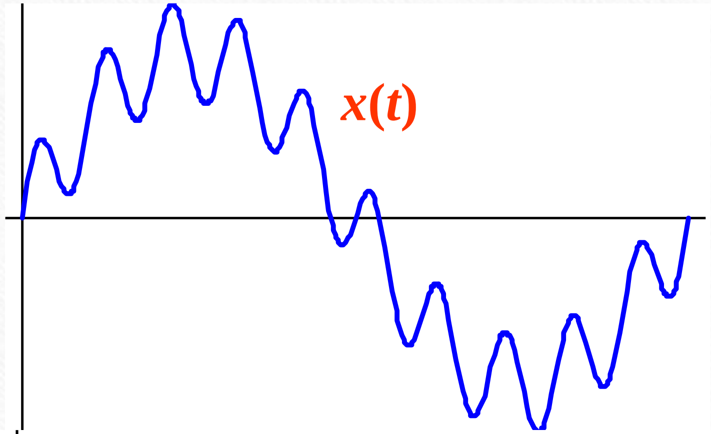
Measurement System Behavior

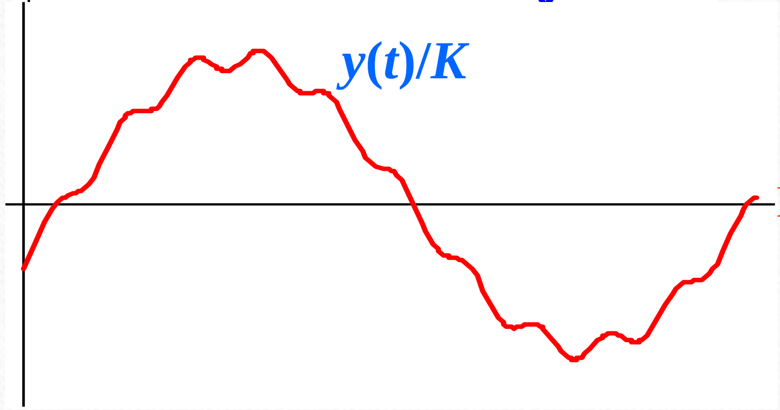


$$m\frac{d^2y}{dt^2}+c\frac{dy}{dt}+ky=c\frac{dx}{dt}+kx$$

First-Order Systems: Frequency Response

Ex: Inadequate frequency response





Suppose we want to measure

$$x(t) = \sin 2t + 0.3 \sin 20t$$

With a first-order instrument whose τ is 0.2 s and static sensitivity K

Superposition concept:

For $\omega = 2$ rad/s: $$B(2\text{ rad/s}) = \frac{K}{\sqrt{0.16+1}} \angle -21.8^{\circ} = 0.93K \angle -21.8^{\circ}$$

For $\omega = 20$ rad/s: $$B(20\text{ rad/s}) = \frac{K}{\sqrt{16+1}} \angle -76^{\circ} = 0.24K \angle -76^{\circ}$$

Therefore, we can write $y(t)$ as

$$y(t) = (1)(0.93K)\sin(2t - 21.8^{\circ}) + (0.3)(0.24K)\sin(20t - 76^{\circ})$$

$$y(t) = 0.93K \sin(2t - 21.8^{\circ}) + 0.072K \sin(20t - 76^{\circ})$$

Dynamic Characteristics

Example: A first order instrument is to measure signals with frequency content up to 100 Hz with an inaccuracy of 5%. What is the maximum allowable time constant? What will be the phase shift at 50 and 100 Hz?

Solution: Define $\left|\dfrac{Q_o(i\omega)}{Q_i(i\omega)}\right| = M(\omega) = \dfrac{K}{\sqrt{\omega^2\tau^2+1}}$

$$\text{Dynamic error} = \frac{M(\omega)-M(0)}{M(0)}\times 100\% = \left(\frac{1}{\sqrt{\omega^2\tau^2+1}}-1\right)\times 100\%$$

From the condition |Dynamic error| < 5%, it implies that $0.95 \le \dfrac{1}{\sqrt{\omega^2\tau^2+1}} \le 1.05$

But for the first order system, the term $1/\sqrt{\omega^2\tau^2+1}$ can not be greater than 1 so that the constrain becomes

$$0.95 \le \frac{1}{\sqrt{\omega^2\tau^2+1}} \le 1$$

Solve this inequality give the range $0 \le \omega\tau \le 0.33$

The largest allowable time constant for the input frequency 100 Hz is

$$\tau = \frac{0.33}{2\pi 100\,\text{Hz}} = 0.52\,\text{ms}$$

The phase shift at 50 and 100 Hz can be found from

$$\phi = -\arctan\omega\tau$$

This give ϕ = -9.33° and = -18.19° at 50 and 100 Hz respectively

Second-Order Systems

In general, a second-order measurement system subjected to arbitrary input, ***x*(*t*)**

$$a_2 \frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t) \quad \Rightarrow \quad \left(\frac{D^2}{\omega_n^2} + \frac{2\zeta}{\omega_n} D + 1 \right) y(t) = Kx(t)$$

$$\frac{1}{\omega_n^2} \frac{d^2 y(t)}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dy(t)}{dt} + y(t) = Kx(t)$$

The essential parameters

$K = \frac{b_0}{a_0}$ = the static sensitivity

$\zeta = \frac{a_1}{2\sqrt{a_0 a_2}}$ = the damping ratio, dimensionless

$\omega_n = \sqrt{\frac{a_0}{a_2}}$ = the natural angular frequency

Second-Order Systems

Consider the characteristic equation $$\frac{1}{\omega_n^2}D^2 + \frac{2\zeta}{\omega_n}D + 1 = 0$$

This quadratic equation has two roots: $$S_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Depending on the value of ζ, three forms of complementary solutions are possible

Overdamped (ζ > 1): $$y_{oc}(t) = C_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}$$

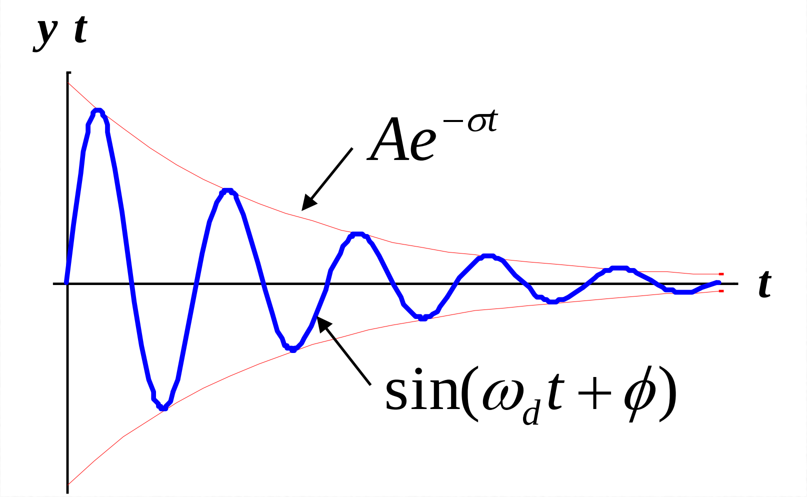
Critically damped (ζ = 1): $$y_{oc}(t) = C_1 e^{-\omega_n t} + C_2 t e^{-\omega_n t}$$

Underdamped (ζ< 1): : $$y_{oc}(t) = C e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1 - \zeta^2}\, t + \Phi\right)$$

Second-Order Systems

Case I Underdamped (ζ< 1):

$$S_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$
$$= \sigma \pm j\omega_d$$

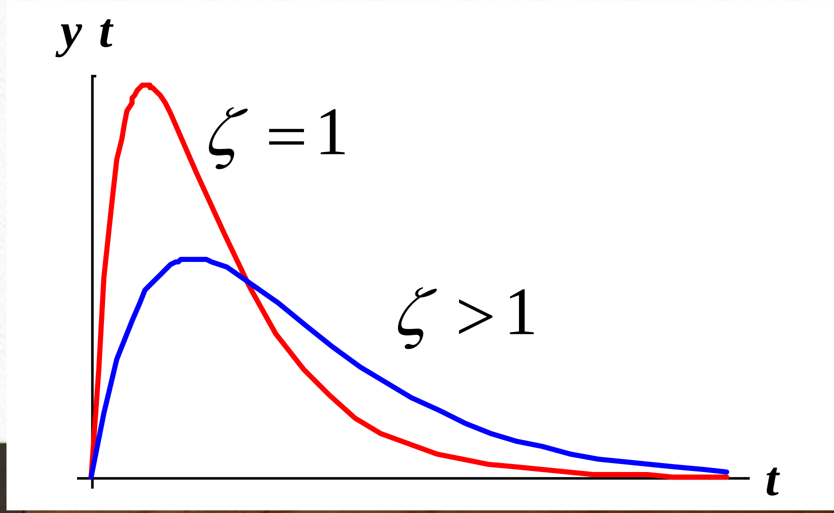


Case 2 Overdamped (ζ > 1):

$$S_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

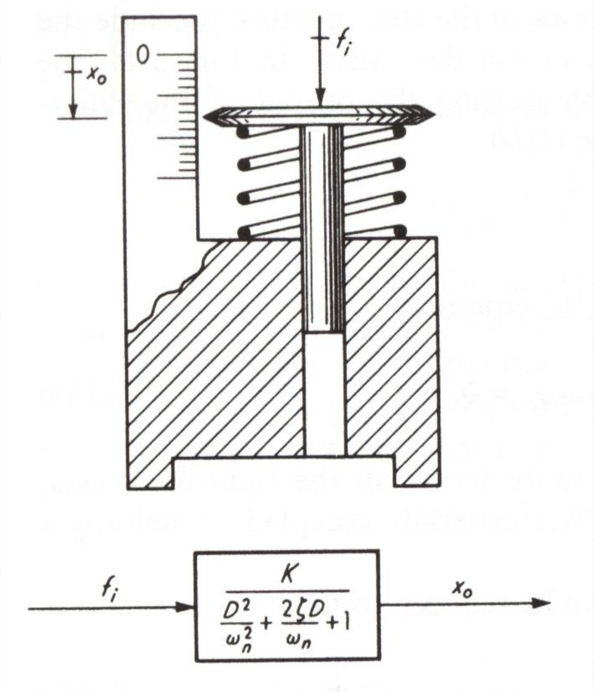
Case 3 Critically damped (ζ = 1):

$$S_{1,2} = -\omega_n$$



Second-order Systems

Example: The force-measuring spring



consider a spring with spring constant K_s under applied force f_i and the total mass ***M***. At start, the scale is adjusted so that $x_o = 0$ when $f_i = 0$;

Σforces=(mass)(acceleration)

$$f_i - B\frac{dx_o}{dt} - K_s x_o = M\frac{d^2 x_o}{dt^2}$$

$$(MD^2 + BD + K_s)x_o = f_i$$

the second-order model:

$$K = \frac{1}{K_s} \quad \text{m/N}$$

$$\omega_n = \sqrt{\frac{K_s}{M}} \quad \text{rad/s}$$

$$\zeta = \frac{B}{2\sqrt{K_s M}}$$

Second-order Systems

For overdamped ($\zeta > 1$) or critical damped ($\zeta = 1$), there is neither overshoot nor steady-state dynamic error in the response.

In an underdameped system ($\zeta < 1$) the steady-state dynamic error is zero, but the speed and overshoot in the transient are related.

Rise time: $t_r = \frac{\arctan(-\omega_d / \delta)}{\omega_d}$

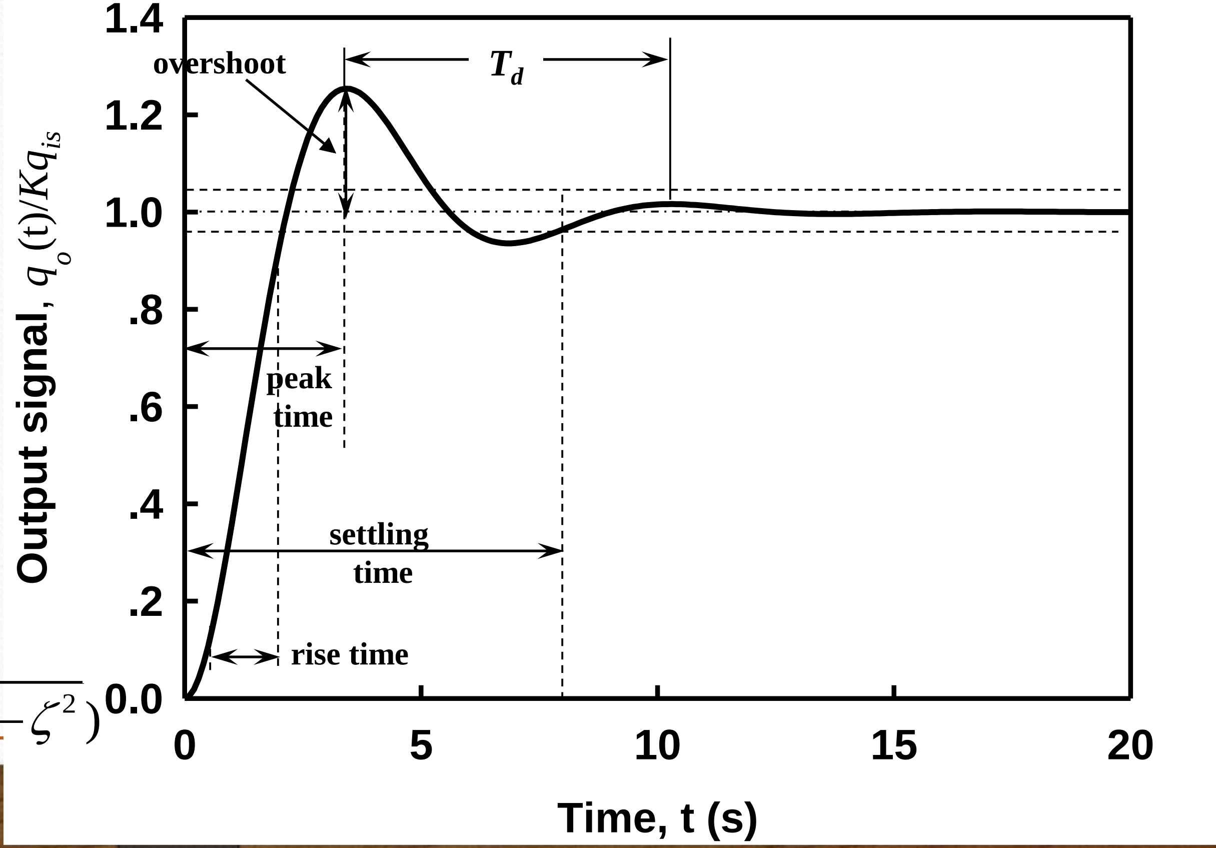
Maximum overshoot: $M_p = \exp\left(-\pi\zeta / \sqrt{1-\zeta^2}\right)$

Peak time: $t_p = \frac{\pi}{\omega_d}$

Resonance frequency: $\omega_r = \omega_n \sqrt{1-2\zeta^2}$

Resonance amplitude: $M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$

where $\delta = \zeta\omega_n, \omega_d = \omega_n\sqrt{1-\zeta^2}$, and $\phi = \arcsin(\sqrt{1-\zeta^2})$



Dynamic Characteristics

Speed of response: indicates how fast the sensor (measurement system) reacts to changes in the input variable. (Step input)

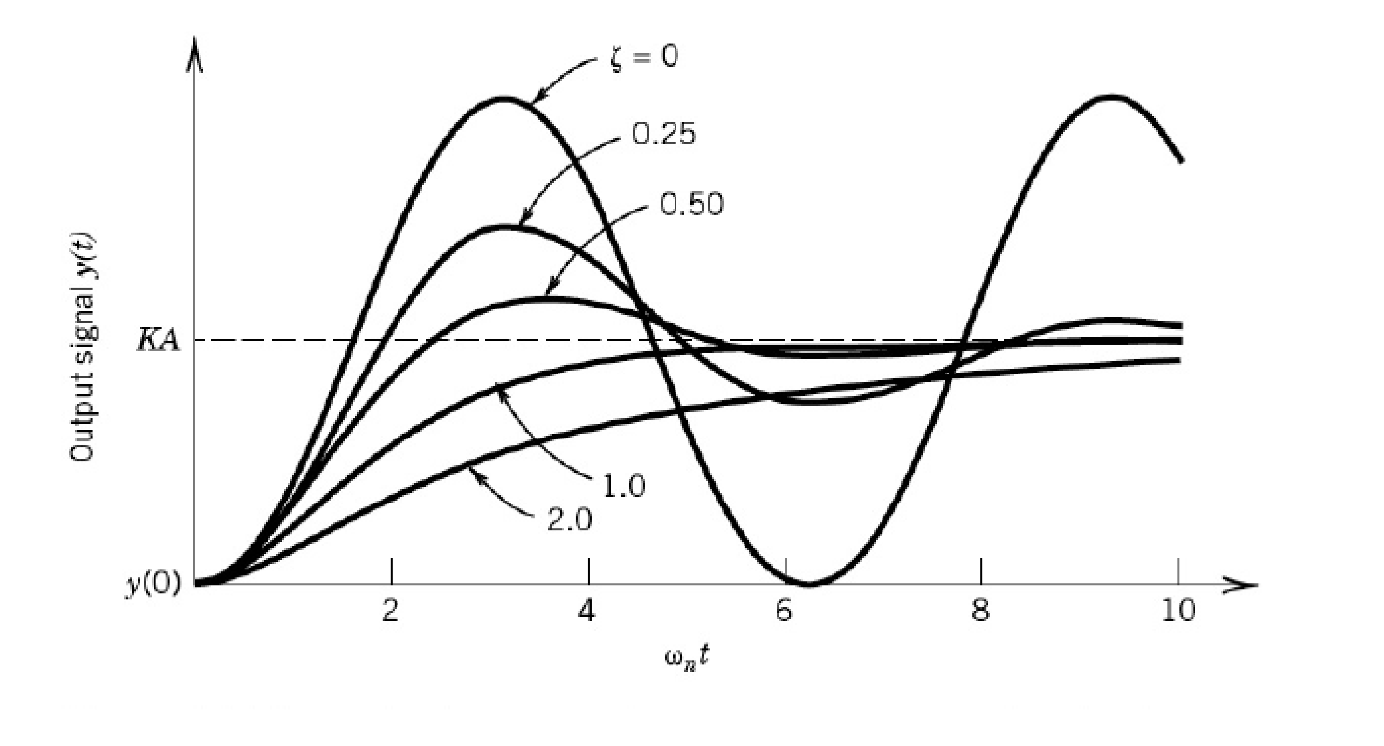
Rise time: the length of time it takes the output to reach 10 to 90% of full response when a step is applied to the input

Time constant: (1^{st} order system) the time for the output to change by 63.2% of its maximum possible change.

Settling time: the time it takes from the application of the input step until the output has settled within a specific band of the final value.

Dead time: the length of time from the application of a step change at the input of the sensor until the output begins to change

Measurement System Behavior



Sensed quantities

- Force/Torque
- Displacement
- Velocity
- Temperature
- vibration
- acceleration
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